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On the refraction of shock waves

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This paper discusses the refraction of plane shock waves in media with arbitrary equations of state. Previous work is reviewed briefly, then a rigorous definition of wave impedance is formulated. Earlier definitions are shown to be unsatisfactory. The impedance is combined with the boundary conditions at the media interface to study both head-on and oblique shock incidence. The impedance determines the nature of the reflected and transmitted waves, their intensities, and the fractions of energy and power that are reflected and transmitted. The refractive index is also defined and determines whether or not a wave will be refracted, and also helps determine whether the wave system will be regular or irregular. The fundamental law of refraction is derived and shown to be a consequence of the fact that an arbitrary point on a shock or an expansion wave follows a ray path of minimum time between any two points on the path. This is a generalization of Fermat's Principle to media that are deformed and convected by the waves propagating through them.

1. Introduction

Consider a longitudinal wave i propagating in a compressible medium and suppose that the wave eventually encounters a second medium, which causes its velocity to change from U_i, to U_t, say. Then by definition i has been refracted if U_t differs from U_i. After refraction i becomes the transmitted wave t, while the second medium becomes the receiving medium (figure 1). The refractive index n of the media is defined as

$$n \equiv \frac{U_i}{U_t}. \tag{1}$$

When $n < 1$, the refraction is slow–fast, and when $n > 1$, it is fast–slow, but there is no refraction when $n = 1$ even if the media differ in composition or in state.

For simplicity it will be assumed that the media are always in contact, and that the interface between them is a plane surface. This means that any convection induced in the media by the passage of the waves must be such that the normal components of the media velocity vectors u_2 and u_t are continuous across their interface. It will also be assumed that the pressure P is continuous. A typical refraction results in the appearance of a reflected wave r as i crosses the interface, so that the media boundary continuity conditions are

$$u_t = u_2, \tag{2}$$

$$P_t = P_2. \tag{3}$$

The wave media are presumed to be contained within a system with well-defined adiabatic boundaries, and they, together with the state parameters of the media,

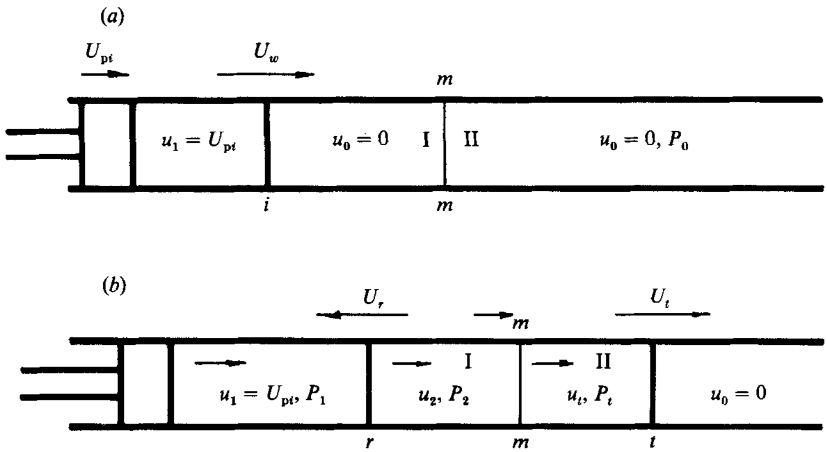

FIGURE 1. Refraction of a normal shock wave at head-on incidence.

namely, the pressure P, and the temperature T, completely define the initial state of the system.

At say time $\tau = 0$, the incident wave i is generated by a suitable boundary disturbance such as a driving or withdrawing piston. A driving piston reduces the volume of the system in unit time so it generates compression waves, which can either be a band of isentropic waves or a shock. In general the withdrawing piston generates only expansion waves. In this paper attention will be limited to those boundaries that cause either steady state or self-similar motions inside the system. Typically, such motions have constant states upstream and downstream of the wave system with no length or time scale associated with the two states. The problem then is to find the flow connecting the two states; it is usually called the Riemann problem (Courant & Friedrichs 1948, p. 181).

In the refraction illustrated in figure 1 the waves are parallel to the interface, so the angle of incidence that i makes to the interface is zero, $\omega_i = 0$. The phenomena become much more complicated when this angle is not zero. In fact many different wave systems have been detected during experiments with gases. Some of them are illustrated in figure 2. For analytical purposes it will be convenient to classify these systems into two types. In the first type the waves are all locally plane and meet at a single point on the interface; these are the *regular* refractions. When i is a shock wave, then it is found that so also is the transmitted wave t, but that the reflected wave may be either a shock r or an expansion e (figure $2a, b$). The natures and intensities of the reflected and transmitted waves, and also the fractions of energy that are reflected and transmitted at the interface, are essentially determined by the wave impedance Z; this quantity will be defined rigorously below. The second type comprises the *irregular* refractions and are simply all those wave systems that fail the definition of the first type. For these refractions one often sees Mach reflections appearing in the incident medium, and when the refraction is also slow–fast ($n < 1$), there is often a precursor shock present in the receiving medium. Generally speaking, the theory is quite successful for the regular refractions but rather unsuccessful for the irregular ones. For simplicity the *undisturbed* interface between any two media will be assumed to be planar. However the piston motion, and the waves generated by it and the refraction, deform the media and set up convection currents in them.

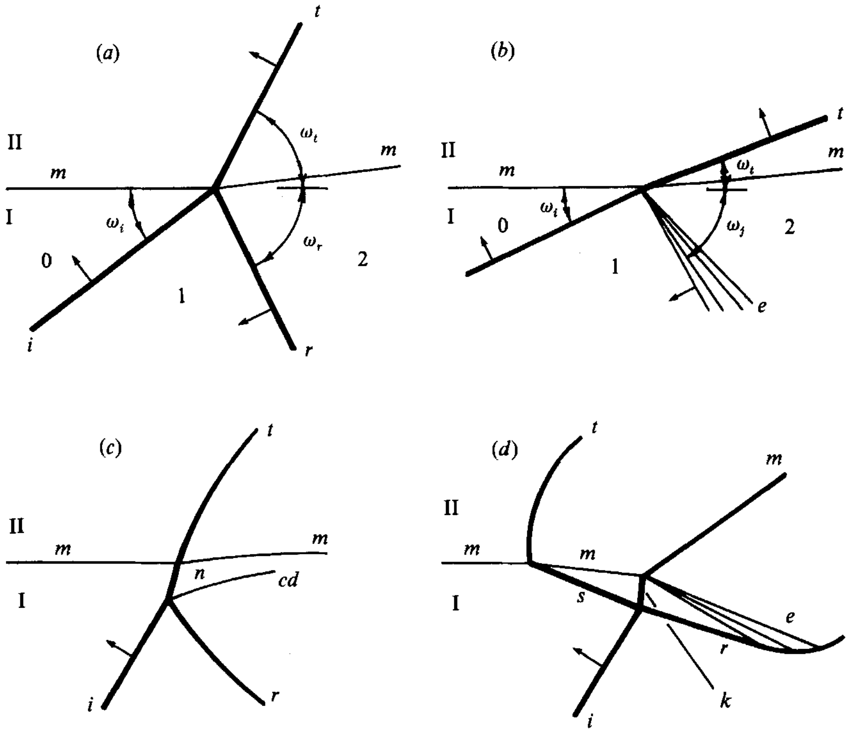

FIGURE 2. Typical regular and irregular shock-wave refraction systems. (*a*) Regular refraction with a reflected shock wave *r*, RRR. (*b*) Regular refraction with a reflected expansion wave *e*, RRE. (*c*) Irregular refraction with a Mach reflection, IRMR. (*d*) Irregular refraction with a free-precursor shock system *ts*, FPR. *i*, incident shock; *t*, transmitted shock; *r*, reflected shock; *e*, expansion wave; *n*, Mach shock; *ts*, transmitted and side shock precursor shocks; *k*, modified shock; *mm*, media interface; I, incident medium; II, transmitting medium; ω_i, angle of incidence; ω_r, angle of reflection; ω_t, angle of transmission.

One consequence is that the media interface, *mm*, is deflected by the passage of the waves, and therefore it does not necessarily lie in a *single* plane (figure 2).

The objective of the present paper is to study the refraction of longitudinal, nonlinear waves which are propagating in compressible media whose equations of state are subject only to the restriction that each medium is in a single phase – although the phases of each medium are not necessarily the same ones. It will be assumed that the media are in local thermodynamic equilibrium (LTE) and that the equilibrium is stable. Violation of this condition leads to a change in the structure of the wave system and the appearance of a contact discontinuity *cd*, such as when a regular refraction changes into an irregular system with a Mach reflection appearing in the initial medium (figure 2*c*).

By definition a wave will be refracted whenever the refractive index n of the medium in which it is propagating changes, that is, $n \neq 1$. The quantities n and Z will be found to be particularly useful for classifying and analysing the various refracting wave systems, in fact for the special case of refraction at a zero angle of incidence $\omega_i = 0$, knowledge of n for the media and Z for each of the waves is tantamount to a complete one-dimensional solution of the problem. For the more general and much more complicated refraction at oblique incidence it is necessary to supplement n and Z with the fundamental law of refraction which will be derived below. The law reduces to Snell's law at the acoustic limit. Then with this extra information it is shown that n is a measure of the capacity of the system to bend i as it passes into the second medium. The quantities n for the media and Z for the waves again amount to a solution, although only for regular refraction. Much more detailed information is required to solve an irregular system. Again, in the interests of simplicity the phenomena that will be discussed will be limited to those that are either in the stationary or pseudostationary (self-similar) state. Naturally this excludes for example oscillatory or accelerating wave systems.

Finally it will be shown that the ray path traced out by any point on a propagating wave is one of the minimum time, which is a generalization of Fermat's Principle to media that are deformed and convected by the passage of the wave. It is found that the refraction law is just the condition that the time has a stationary value, and the fact that it is also a minimum is a necessary condition that the local thermodynamic equilibrium should be stable.

2. Previous work

Experiments with shock waves refracting in gases have been done by Jahn (1956), Abdel-Fattah, Henderson & Lozzi (1976), and Abdel-Fattah & Henderson (1978*a*, *b*). More recently, Reichenbach (1985) has done experiments with shocks refracting at thermal layers, and Haas & Sturtevant (1987) with refraction by gaseous cylindrical and spherical inhomogeneities. Earlier, Dewey (1965) reported on precursor shocks from large-scale explosions in the atmosphere. Some multiphase experiments have also been done: Sommerfeld (1985) has studied shocks refracting from pure air into air containing dust particles, while Gvozdeava *et al.* (1986) have experimented with shocks passing from air into a variety of foam plastics.

Early work of the theory or regular refraction was done by Taub (1947) and Polachek & Seeger (1951). Later, Henderson (1966) extended this work to irregular refraction with the help of polar diagrams. He also generalized the definition of shock-wave impedance given by Polachek & Seeger for the refraction of normal shocks (Henderson 1970), but it will be shown below that these definitions are unsatisfactory. Flores & Holt (1982) have studied the refraction of shock waves at air–water interfaces. The Whitlam theory has been applied to shocks propagating in non-uniform media by Catherasoo & Sturtevant (1983), and the results compared with the Abdel-Fattah & Henderson data.

More recently numerical technique has been applied to refraction problems: Picone *et al.* (1984) have studied the vorticity generated when a shock is refracted by a flame. They compared their results with Markstein's (1964) experiments and claim to have reproduced most of his experimental observations. Picone *et al.* (1986) have studied the Haas & Sturtevant experiments at air/helium and air/freon, cylindrical, and spherical interfaces. Fry & Book (1986) have considered refraction at heated layers, and Glowacki *et al.* (1986) have studied refraction at high-speed sound layers. Sigimura, Tokita & Fujiwara (1984) have studied refraction in a liquid–bubble system.

3. Wave impedance, Z

3.1. *Normal-shock impedance*

The wave impedance is defined in general terms as the force per unit area (pressure, stress) which must be applied to a medium in order to impart a unit particle velocity to some part of the medium. Consider a piston driving into a compressible medium at a velocity U_{pi}, and generating a normal shock wave i with wave velocity U_i, figure 1, then by definition,

$$Z_i \equiv \pm \frac{P_1 - P_0}{U_{pi}} = \frac{P_1 - P_0}{u_1 - u_0}, \tag{4}$$

were P is the pressure, u is the particle velocity and the subscripts 0, 1 refer to conditions upstream and downstream of the shock respectively. A shock wave is compressive, so $P_1 - P_0 > 0$, then $Z_i > 0$ if the piston drives in the positive x-direction, and $Z_i < 0$ if in the negative direction. The quantities $(P_1 - P_0)$, and $(u_1 - u_0)$ are discontinuous across the shock, but they become infinitesimal at the acoustic limit, and from (4) one then recovers the acoustic 'Ohm's law', $\Delta P = Z_a \Delta u$, where Z_a is the acoustic impedance. If the definition were to be extended to oscillatory motion then Z_i would have to be defined in terms of the complex pressure and complex velocity (Kinsler *et al.* 1982, Chapter 6), but in the interests of simplicity the definition is restricted to the Riemann-type problems for stationary and pseudostationary (self-similar) motion where $U_{pi} = u_1 - u_0$ is constant.

It is convenient to express Z_i entirely in terms of variables of state, which requires that $(u_1 - u_0)$ be eliminated from (4). This can be done with the help of the continuity and momentum equations, which for the above restrictions become

$$u_1/v_1 = u_0/v_0, \tag{5}$$

$$P_1 + u_1^2/v_1 = P_0 + u_0^2/v_0, \tag{6}$$

where v is the specific volume. These give

$$u_0^2 = -v_0^2 \frac{P_1 - P_0}{v_1 - v_0}, \tag{7}$$

$$u_1^2 = -v_1^2 \frac{P_1 - P_0}{v_1 - v_0}, \tag{8}$$

and therefore

$$u_1 - u_0 = \pm[-(v_1 - v_0)(P_1 - P_0)]^{\frac{1}{2}}, \tag{9}$$

and on substituting into (4),

$$Z_i = \pm\left[-\left(\frac{P_1 - P_0}{v_1 - v_0}\right)\right]^{\frac{1}{2}}, \tag{10}$$

or alternatively using (7) and (8),

$$Z_i = \pm u_0/v_0 = \pm \rho_0 u_0, \tag{11}$$

$$Z_i = \pm u_1/v_1 = \pm \rho_1 u_1, \tag{12}$$

where ρ is the density. Only the definition of Z_i and the equations of continuity and momentum have been used to derive (10), (11) and (12), so the equations are valid for any Rayleigh process, such as a normal shock wave, in a medium with an arbitrary equation of state, $P = P(s, v)$ say, where s is the entropy. However we have assumed that the system is adiabatic and that the medium is in stable, local thermodynamic equilibrium. These conditions are nearly always satisfied when the medium is in a single phase, Bethe (1942). In these circumstances his 'central (theorem)' guarantees a unique solution to the Hugoniot equations (5)–(9), given only the initial state of the medium and U_{pi}. One can calculate for example $(P_1 - P_0)$ and then Z_i. Conversely if the initial state U_{pi} and Z_i are given then this amounts to a solution of the equations.

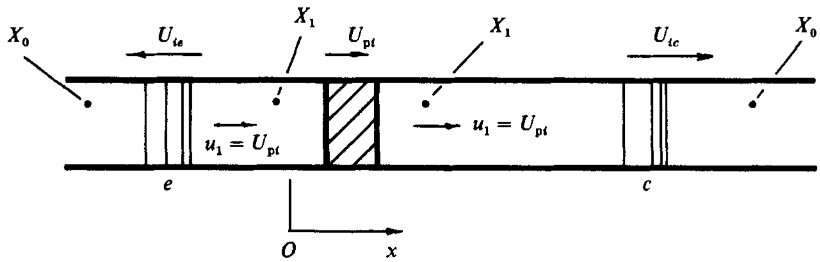

FIGURE 3. Isentropic compression and expansion waves generated by a slowly accelerating piston.

It will be noticed from (10) that $-Z_i^2$ is the slope of the Rayleigh line, and that it is also proportional to the average adiabatic bulk modulus, $\Delta P/\Delta v$. Furthermore, by (11) Z_i is proportional to the shock velocity U_i, because in shock coordinates $-U_i = u_0$, but this means by (7)–(11) that Z_i also depends on the amplitude $(P_1 - P_0)$, or $(v_1 - v_0)$ of the shock. Therefore Z_i depends not only on the properties of the medium, such as ρ_0, but also on the boundary conditions of the system, namely U_{pi}, which determine the wave speed and amplitude for a given state. By contrast the wave acoustic impedance is a property of the medium and does not depend on the wave amplitude.

The special case when the medium is a perfect gas

When a normal shock propagates through a perfect gas, one can use the Rankine–Hugoniot equations (Ames 1953) to obtain a relationship between P_1/P_0 and v_1/v_0, and on substituting it into (10) we get

$$Z_i = \pm \rho_0 a_0 \left[\frac{1}{2\gamma}\left((\gamma-1)+(\gamma+1)\frac{P_1}{P_0}\right)\right]^{\frac{1}{2}}, \tag{13}$$

where γ is the ratio of specific heats. At the acoustic limit when $P_1/P_0 \rightarrow 1$, then $Z_i \rightarrow \pm \rho_0 a_0$, as it should. Equation (13) can be compared with the Polachek & Seeger (1951) definition, namely,

$$Z_{PS} \equiv \pm \frac{1}{a_0}\left[\gamma\left((\gamma+1)+(\gamma-1)\frac{P_0}{P_1}\right)\right]^{\frac{1}{2}}. \tag{14}$$

At the acoustic limit, $Z_{PS} \rightarrow \pm \rho_0 a_0 \sqrt{2/P_0}$ or $Z_{PS} \rightarrow \pm (2\gamma/RT_0)^{\frac{1}{2}}$, where R is the gas constant and T_0 the temperature of the undisturbed gas. Thus Z_{PS} is a function of T_0 only, and not of *both* $a_0(T_0)$, and ρ_0, as required if Z_{PS} is to be consistent with the acoustic impedance, so their definition is unsatisfactory.

3.2. *The slowly and uniformly accelerating piston*

Suppose that after a slow and uniform acceleration, a piston reaches a constant velocity U_{p_i}, figure 3. Then there will be an unsteady band of compression waves on the driving side and a similar band of expansion waves on the withdrawing side. It will be assumed that the waves are isentropic, and that $x = 0$ when $\tau = 0$. Now choose a point x which at the time τ is ahead of the right-moving waves. If x moves at the same speed a_0 as the front wave, then it will always be ahead of it (so long as a shock has not formed). Introducing the self-similar coordinate $X \equiv x/\tau$, then $X_0 = x_0/\tau = a_0$ will be chosen so that it is always ahead of the compression waves. Similarly,

$X_1 \equiv x_1/\tau = U_{pi}$ will be chosen so that it is always between the piston and the last wave. On the withdrawing side one can again choose X_0, X_1, to be always on either side of the wave band. Then for a self-similar system both P_1 and $u_1 = U_{pi}$ do not change with time at X_1, and neither do P_0 nor u_0 at X_0, even though the flow is unsteady. It follows by (4) that Z_i is also constant. So this is also a Riemann problem. For the special case of isentropic waves in a perfect gas,

$$-u_1 \pm \frac{2}{\gamma-1} a_1 = -u_0 \pm \frac{2}{\gamma-1} a_0,$$

and so

$$-(u_1-u_0) = \pm\frac{2a_0}{\gamma-1}\left(1-\frac{a_1}{a_0}\right) = \pm\frac{2a_0}{\gamma-1}\left[1-\left(\frac{P_1}{P_0}\right)^{(\gamma-1)/2\gamma}\right], \tag{15}$$

and on substituting into (4),

$$Z_i = \pm\frac{\gamma-1}{2\gamma}\rho_0 a_0 \frac{P_1/P_0-1}{1-(P_1/P_0)^{(\gamma-1)/2\gamma}}, \tag{16}$$

where we have also used $a_0^2 = \gamma P_0/\rho_0$. It is easily verified that in the acoustic limit $Z_i \to \pm\rho_0 a_0$, as it should.

4. Refraction of a normal shock wave at an interface between two media

4.1. *Definition of the reflection and transmission coefficients*

Consider a plane shock wave i propagating in a medium with an arbitrary equation of state, and which is initially at rest, figure 1. Suppose that the shock then encounters a second medium, also initially at rest, but with a different impedance to the first one. It will be assumed that the interface mm between the two media is plane and parallel to the shock, so that i makes a head-on collision with it. Then i may be refracted at the interface and give rise to a transmitted shock t in the second medium, and a reflected wave in the first medium which may either be a shock r, or an expansion e.

The pressure reflection and transmission coefficients for this system are defined as follows:

$$R \equiv \frac{P_2-P_1}{P_1-P_0}, \tag{17}$$

$$T \equiv \frac{P_t-P_0}{P_1-P_0}. \tag{18}$$

The shock intensity I is defined to be the average power flux through unit area normal to the direction of propagation. For example, for i,

$$I_i \equiv (P_1-P_0)(u_1-u_0) = (P_1-P_0)U_{pi}, \tag{19}$$

with similar definitions for I_r and I_t. Combining with (4),

$$I_i = \pm\frac{(P_1-P_0)^2}{Z_i}, \tag{20}$$

and again also for I_r, and I_t. The intensity reflection and transmission coefficients are defined as

$$R_I \equiv \frac{I_r}{I_i} = \left|\frac{(P_2-P_1)^2}{(P_1-P_0)^2}\frac{Z_i}{Z_r}\right| = R^2\left|\frac{Z_i}{Z_r}\right|, \tag{21}$$

$$T_I = \frac{I_t}{I_i} = \left|\frac{(P_t-P_0)^2}{(P_1-P_0)^2}\frac{Z_i}{Z_t}\right| = T^2\left|\frac{Z_i}{Z_t}\right|. \tag{22}$$

The power transmitted along a stream tube of cross-sectional area A is AI, so the power reflection and transmission coefficients are defined as

$$R_\pi \equiv \frac{A_r}{A_i}\frac{I_r}{I_i} = \frac{A_r}{A_i}\frac{(P_2-P_1)^2}{(P_1-P_0)^2}\left|\frac{Z_i}{Z_r}\right| = \frac{A_r}{A_i}R^2\left|\frac{Z_i}{Z_r}\right|, \tag{23}$$

$$T_\pi \equiv \frac{A_t}{A_i}\frac{I_t}{I_i} = \frac{A_t}{A_i}\frac{(P_t-P_0)^2}{(P_1-P_0)^2}\left|\frac{Z_i}{Z_t}\right| = \frac{A_t}{Z_i}T^2\left|\frac{Z_i}{Z_t}\right|. \tag{24}$$

4.2. *The boundary conditions at the interface*

The boundary conditions that must be satisfied at the interface are (2) and (3), which may be more conveniently written as

$$(P_2-P_1)+(P_1-P_0) = (P_t-P_0), \tag{25}$$

$$(u_1-u_0)+(u_2-u_1) = (u_t-u_0), \tag{26}$$

and so

$$U_{pi}+U_{pr} = U_{pt}, \tag{27}$$

where the last equation is written in terms of the piston (particle) velocities. Dividing (25) by (27),

$$\frac{(P_2-P_1)+(P_1-P_0)}{(u_1-u_0)+(u_2-u_1)} = \frac{P_t-P_0}{u_t-u_0} = Z_t, \tag{28}$$

and then eliminating (u_1-u_0), etc. with Z_i, etc. results in

$$R = \frac{Z_r}{Z_i}\frac{Z_t-Z_i}{Z_r-Z_t}, \tag{29}$$

but when we divide (25) by (P_1-P_0) we get,

$$T = 1+R, \tag{30}$$

and so

$$T = \frac{Z_t}{Z_i}\frac{Z_i-Z_t}{Z_t-Z_r}. \tag{31}$$

The shock i can be imagined as being generated by a piston which impulsively acquires the velocity U_{pi} at $\tau = 0$, and in the positive x-direction, $U_{pi} > 0$. Since a driving piston will compress the medium $(P_1-P_0) > 0$, and $Z_i > 0$. After refraction the piston begins driving the second medium, also in the positive direction, and compresses it so $Z_t > 0$ whenever $Z_i > 0$. The conclusion is valid irrespective of the nature of the reflected wave. By (31) we now have $T > 0$. If the piston drives in the negative x-direction $Z_i < 0$ and $Z_t < 0$, but again $T > 0$. By (29), when $|Z_t| > |Z_i| > 0$, then $R > 0$, and the reflected wave is a shock, but when $|Z_i| > |Z_t| > 0$, then $R < 0$, and it is an expansion. When the media have the same impedance $Z_t = Z_i$, then $R = 0$, and the reflected wave is a Mach line or acoustic degeneracy, $P_2 = P_1$, and there is *complete transmission* $T = 1$, by (31). It is concluded that the sign of Z_t is always the same as Z_i, and T is always positive, while R may be positive or negative depending on whether $|Z_t| > |Z_i|$ or vice versa, or R will be zero when $|Z_t| = |Z_i|$.

The left-hand side of (28) may be defined to be the impedance for the combined incident and reflected waves Z_{ir}, or Z_{ie}. Thus is it concluded that the system responds to the driving piston by adjusting the wave impedances of the two media to be equal, $Z_{ir} = Z_t$, or $Z_{ie} = Z_t$.

4.3. *The refraction limits*

4.3.1. *The acoustic limit*

At this limit $Z_t \to r_t \equiv \rho_t a_t$, $Z_r \to Z_i \to r_i \equiv \rho_0 a_0$, and (29) and (31) reduce to the well-known acoustic formulas (Kinsler *et al.* 1982)

$$R = \frac{1 - r_i/r_t}{1 + r_i/r_t}, \quad T = \frac{2}{1 + r_i/r_t}. \tag{32}$$

These equations are symmetrical in r_i and r_t, so R and T remain the same irrespective of whether i passes from the first medium into the second, or vice versa. This is the *principle of acoustic reciprocity*, but it cannot be extended to shock waves because (29) and (31) are not symmetrical in Z_i and Z_t

4.3.2. *The rigid limit*

If the impedance of the second medium increases without limit, $Z_t \to \infty$, then it will become a rigid body. We shall assume that Z_i and Z_r will remain finite and non-zero as this happens, then

$$R \to \frac{Z_r}{Z_i} > 0, \quad T \to 1 + \frac{Z_r}{Z_i} > 0, \quad R_I \to \frac{Z_r}{Z_i}, \quad T_I \to 0.$$

During head-on refraction there is no change in the cross-sectional area, so R_π and T_π will be the same as R_I and T_I. We see that although a shock wave can penetrate a rigid body there is no energy transmitted into it; this is *total reflection*. The reflected wave is always a shock at this condition because $R > 0$.

The analysis can be taken further for the special case when the first medium is a perfect gas. The inverse shock strengths $\xi_i \equiv P_0/P_1$, and $\xi_r \equiv P_1/P_2$, are now related by the von Neumann (1943) formula

$$\xi_r = \frac{(\gamma-1)+(\gamma+1)\,\xi_i}{(3\gamma-1)-(\gamma-1)\,\xi_i}, \tag{33}$$

from which

$$R = \frac{2\gamma}{(\gamma-1)+(\gamma+1)\,\xi_i}, \tag{34}$$

so

$$1 \leqslant R \leqslant \frac{2\gamma}{\gamma-1} \quad \text{for } 1 \geqslant \xi_i \geqslant 0,$$

where the left-hand limit occurs for an acoustic wave $\xi_i = 1$, and the right-hand limit when the shock is infinitely strong, $\xi_i = 0$. Furthermore,

$$2 \leqslant T \leqslant \frac{3\gamma-1}{\gamma-1},$$

so *the intensity of a shock which penetrates a rigid body is at least double that of the incident shock.* However no energy is transmitted into the body because $T_\pi = T_I = 0$.

4.3.3. *The compliant limit*

In this case $Z_t \to 0$, and once again we assume that Z_i and Z_r remain non-zero and finite; then $R \to -1$, and $T \to 0$. The reflected wave is now an expansion and the transmitted shock is reduced to an acoustic degeneracy, $P_t = P_0$. In acoustics it is sometimes called the *pressure release condition*, but we shall call it the *compliant limit*. It occurs for example if a normal shock propagating inside a tube encounters an open end of the tube; since $T_\pi = T_I = 0$, no energy is transmitted into the second medium, so it is also the condition of *total internal reflection*.

Furthermore, if $Z_i \to \infty$ and $Z_r \to \infty$ while Z_t remains finite then $R \to -1$ and $T \to 0$. So a shock propagating in a medium of infinite impedance is also totally internally reflected if it encounters an interface with another medium of finite impedance. Thus extreme impedance mismatch between the media greatly attenuates the transmission of energy and power into the receiving medium, $T_\pi \to T_I \to 0$.

4.4. *The exact theory of normal refraction in a perfect gas*

When the two media satisfy the Bethe conditions described in §3.1, and when their initial states, and U_{pi} are given then the Rankine–Hugoniot equations together with the boundary conditions (25) and (27) provide a complete one-dimensional solution to the normal-shock refraction problem. In particular one can obtain the three impedances Z_i, Z_r, and Z_t. Conversely if the initial states, U_{pi}, and the three impedances are given then this amounts to a solution of the equations, because there are five equations, namely (25), (27), and three impedance definitions (4) for the five unknown quantities U_{pr}, U_{pt}, $(P_1-P_0)(P_2-P_1)$, and (P_t-P_0).

For the special case of a perfect gas, this problem can be reduced to the solution of a single polynomial equation of degree 4. It is only necessary to substitute the Rankine–Hugoniot equations into (25) and (27). If $x \equiv P_1/P_0$, and $y \equiv P_t/P_0$, then after a tedious calculation one can obtain the polynomial in a form that is easy to compute, namely,

$$\begin{aligned} &a_t^4\gamma_i^2(y-1)^2[(\gamma_i-1)+(\gamma_i+1)x]^2[(\gamma_i+1)y+(\gamma_i-1)x]^2 \\ &-2a_t^2a_i^2\gamma_t\gamma_i[(\gamma_i-1)+(\gamma_i+1)x][(\gamma_i+1)y+(\gamma_i-1)x] \\ &\times\{[(\gamma_i+1)+(\gamma_i-1)x](y-x)^2+(x-1)^2[(\gamma_i+1)y+(\gamma_i-1)x]\} \\ &\times[(\gamma_t-1)+(\gamma_t+1)y]+a_i^4\gamma_t^2[(\gamma_t-1)+(\gamma_t+1)y]^2 \\ &\times\{[(\gamma_i-1)-(3\gamma_i-1)x]x+[(\gamma_i+1)+(\gamma_i-1)x]y\}=0, \end{aligned} \tag{35}$$

where a_i, a_t are the undisturbed speeds of sound in the incident and transmitting media, and γ_i, γ_t the corresponding ratios of specific heats. When these quantities and x are known then y can be calculated.

5. Refraction of an oblique shock wave at an interface between two media

5.1. *The boundary conditions and the intensity coefficients*

Once more there is an incident i and transmitted t shock, and a reflected wave that can be either a shock r (RRR) or an expansion r (RRE), figures 2(*a*, *b*). There is again continuity in the pressure across the media interface, so (3), (25), and (30) remain valid, but because the waves are no longer parallel to the interface (27) must be changed to

$$U_{pi}\cos\beta_i+U_{pr}\cos\beta_r=U_{pt}\cos\beta_t, \tag{36}$$

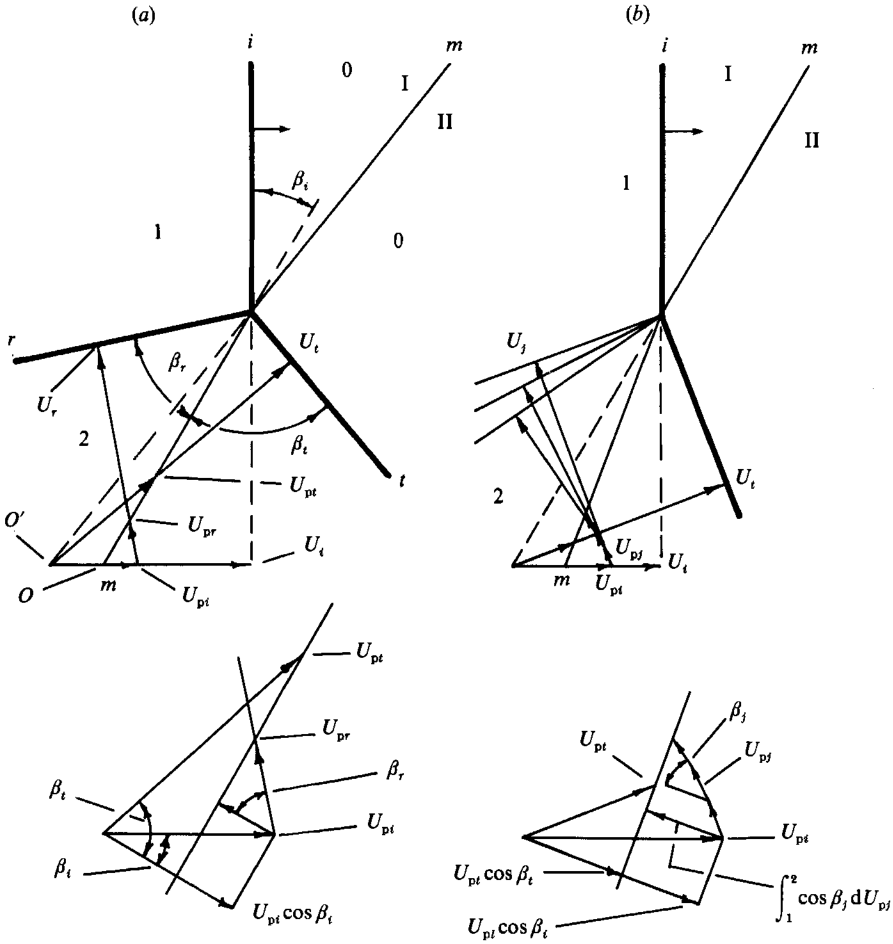

FIGURE 4. Regular refractions illustrating the components of the particle velocity normal to the disturbed interface. (a) RRR; (b) RRE.

where $\beta_i, \beta_r, \beta_t$ are the wave angles of i, r, and t, measured with respect to the *deflected* interface, figure 4. Equation (36) means that the velocity components of the media which are normal to the deflected interface are continuous across it. Now using the impedances to eliminate the piston velocities from (36),

$$Z_r Z_t (P_1 - P_0) \cos \beta_i + Z_i Z_t (P_2 - P_1) \cos \beta_r = Z_i Z_r (P_t - P_0) \cos \beta_t$$

and so
$$Z_r Z_t \cos \beta_i + R Z_i Z_t \cos \beta_r = T Z_i Z_t \cos \beta_t, \tag{37}$$

and eliminating T between (30) and (37) we get

$$R = \frac{Z_r [Z_i \cos \beta_t - Z_t \cos \beta_i]}{Z_i [Z_t \cos \beta_r - Z_r \cos \beta_t]}, \tag{38}$$

which reduces to the Rayleigh formula at the acoustic limit,

$$R \to \frac{r_t \cos \beta_i - r_i \cos \beta_t}{r_t \cos \beta_i + r_i \cos \beta_t}, \tag{39}$$

as may be easily verified. Substituting (38) into (30) gives

$$T = \frac{Z_t[Z_i \cos\beta_r - Z_r \cos\beta_i]}{Z_i[Z_t \cos\beta_r - Z_r \cos\beta_t]}. \tag{40}$$

5.2. *The fundamental law of shock-wave refraction*

If a regular wave system is to retain the same structure as it propagates, then all of its waves must proceed at the same velocity along any straight trajectory path that passes through the wave confluence point. If one takes the path that passes through the confluence and coincides with the disturbed interface, then one must have

$$U = \frac{U_i}{\sin\beta_i} = \frac{U_r}{\sin\beta_r} = \frac{U_t}{\sin\beta_t}, \tag{41}$$

where U_i, U_r, and U_t are the wave velocities of i, r, and t, and all are measured with respect to the origin O, which is at rest with respect to the undisturbed media upstream of the wave system, figure 4. If we wish to take the path coinciding with the undisturbed interface, then the velocities are measured with respect to the new origin O', and the wave angles are now ω_i, ω_r, and ω_t, figure 2. Equation (41) is one form of the fundamental law of shock refraction; it evidently reduces to Snell's law at the acoustic limit and the velocities then reduce to the corresponding acoustic-wave velocities.

5.3. *The wave refractive index*

From (11),
$$Z_i = \pm\rho_i U_i, \quad Z_t = \pm\rho_t U_t, \tag{42}$$

and remembering that Z_t always has the same sign as Z_i we have

$$\frac{Z_i}{Z_t} = \frac{\rho_i U_i}{\rho_t U_t}. \tag{43}$$

But, from the refraction law,

$$\frac{U_i}{\sin\beta_i} = \frac{U_t}{\sin\beta_t}, \tag{44}$$

therefore
$$n \equiv \frac{U_i}{U_t} = \frac{v_i Z_i}{v_t Z_t} = \frac{\sin\beta_i}{\sin\beta_t}. \tag{45}$$

When $n > 1$, $\beta_i > \beta_t$, and the transmitted shock t is bent away from a normal to the disturbed interface. Conversely, when $n < 1$, $\beta_i < \beta_t$, and t will be bent towards the normal. The wave will not be refracted when $n = 1$, because then $\beta_t = \beta_i$. So we see that n is a measure of the capacity of the system to bend or refract a wave.

5.3.1. *The angle of intromission*

There will be no reflected wave when $Z_t = Z_i$, and (45) gives

$$\rho_i U_i = \rho_t U_t, \tag{46}$$

that is, the mass flux through both shocks is equal. However, (46) is not necessarily an identity so in general $n \neq 1$ and the shock will be refracted. Typically the condition appears during the regular refraction transition, RRE $\rightleftharpoons$ RRR, in which a reflected expansion continuously degenerates to a Mach line and then strengthens into a reflected shock, or vice versa, under a continuous change of a system parameter, such as the angle of incidence of i. Examples have been given by Henderson (1966, p. 623), Abdel-Fattah *et al.* (1976, p. 164), and Abdel-Fattah & Henderson (1978*a*, pp. 18, 22; 1978*b*, p. 82). Although there is no reflected wave when $Z_t = Z_i$, nonetheless in general $n \neq 1$, $\beta_t \neq \beta_i$, and the shock will be refracted at the interface. Since $Z_r = 0$ at this condition, then (38) and (40) give $R = 0$ and $T = 1$, which is again the total transmission condition. In acoustics the angle of incidence at which it occurs is called the *angle of intromission* and we shall use the same term. We can further refine these results as follows.

5.3.3. *The critical angle*

We may write

$$\cos\beta_t = (1-\sin^2\beta_t)^{\frac{1}{2}} = (1-n^{-2}\sin^2\beta_i)^{\frac{1}{2}}. \tag{47}$$

(i) When $n > 1$, β_t is real and $\beta_i > \beta_t$ as before.

(ii) When $n < 1$, and $\beta_i < \beta_c$, where β_c is the critical angle defined such that the transmitted shock makes a glancing incidence to the deflected interface $\beta_t = \frac{1}{2}\pi$, and

$$\sin\beta_c = U_i/U_t, \tag{48}$$

then β_t is still real, but $\beta_i < \beta_t$. At the critical condition t is normal to both the disturbed and the undisturbed interface, because $\beta_t = \frac{1}{2}\pi$, so the interface is everywhere in a single plane.

(iii) If $n < 1$, and $\beta_i > \beta_c$, then $\cos\beta_t$ is pure imaginary. We shall find below that the regular shock system then breaks up into an irregular system with some form of precursor shock, figure 2(*d*). The breakup of the system violates the refraction law for the regular system, but it is immediately re-established in a different form for the irregular system.

5.4. *The wave impedance for an oblique shock*

5.4.1. *Regular refraction with a reflected shock*, RRR

Suppose the impedance for an oblique shock, for example i is redefined as follows:

$$Z_i \equiv \frac{P_1 - P_0}{U_{pi}\cos\beta_i}, \tag{49}$$

and similarly for r and t. If these new definitions are substituted into (38) and (40), then the R and T for oblique refraction are reduced to the R and T coefficients for normal refraction (29) and (31), as may be verified by inspection. One may also redefine the oblique-wave intensity as

$$I_i \equiv (P_1 - P_0)\,U_{pi}\cos\beta_i, \tag{50}$$

and again with similar expressions for I_r and I_t. Then the oblique refraction intensity coefficients will also reduce to the normal ones, (21) and (22), and similarly for the power coefficients. It also follows from (49) and from the discussion of §4.4 that when the initial conditions of the media together with U_{pi}, β_i, and the impedances of the three waves are given then this amounts to a solution of the problem of regular refraction. Finally, (49) corrects the Henderson (1970) definition, which suffers from a similar defect to the Polachek & Seeger definition.

5.4.2. *Regular refraction with a reflected expansion* RRE

It is a little more difficult to extend the theory to a regular refraction with a reflected expansion (RRE), because there is no single wave angle β_e for the expansion but a continuum of values. Each expansion wavelet is associated with an infinitesimal withdrawing piston velocity dU_p, and for the jth wavelet this is dU_{pj} which occurs at the angle β_j measured with respect to the disturbed interface. The projection of the dU_{pj} vector onto a normal to the interface is $dU_{pj} \cos\beta_j$, so for the entire expansion wave the total normal vector component is $\int_1^2 \cos\beta_j \, dU_{pj}$ (figure 4). Although difficult to evaluate analytically, the integral is easy enough numerically. Now all we have to do to incorporate an RRE refraction into the theory is to replace the second term of (36) with the integral and define the wave impedance for an expansion fan as

$$Z_e \equiv (P_2 - P_1) \Big/ \int_1^2 \cos\beta_j \, dU_{pj}. \tag{51}$$

By these means both of the oblique regular refraction systems RRR and RRE can be reduced to the corresponding head-on incidence refractions, and the conclusions about the acoustic, the rigid, and the compliant limits remain valid for the oblique systems. The refraction law is of course

$$U = \frac{U_i}{\sin\beta_i} = \frac{U_t}{\sin\beta_t} = \frac{U_j}{\sin\beta_j}, \tag{52}$$

where $U_j = a_j$, is the local speed of sound.

5.5. *The numerical solution of the regular-refraction problem*

When a regular refraction has a reflected shock (RRR), its solution(s) requires the boundary conditions at the interface, (25) and (36), the refraction law (41), the Rankine–Hugoniot equations for all the shocks, and the equation of state. For a perfect gas, all of these equations are reducible to a single polynomial equation of twelfth degree in which the unknown variable is the strength P_t/P_0 of the shock t. The polynomial coefficients are given in the Appendix in a form suitable for computation; in fact a programmable calculator with about 2 or 3 K is enough to handle it.

Closed-form expressions are not available for a regular refraction with a reflected expansion (RRE), but numerical solutions for a perfect gas present no difficulty. The Prandtl–Meyer equation replaces the Rankine–Hugoniot equation for the reflected wave. The theory of RRR and RRE is in good agreement with the experiments of Jahn and Abdel-Fattah *et al.*, as they have shown.

6. The minimum-time principle

6.1. *Regular refraction*

A ray path is traced out by an arbitrary point on a propagating wave; it will now be shown that the path is one of minimum time, at least for stationary and pseudostationary systems. This will extend Fermat's principle from acoustics to shock and expansion waves. The derivation differs from acoustics in that the deflection of the interface by the waves must be taken into account. We demonstrate the derivation for the arbitrary point, S, on the incident shock, as it proceeds along

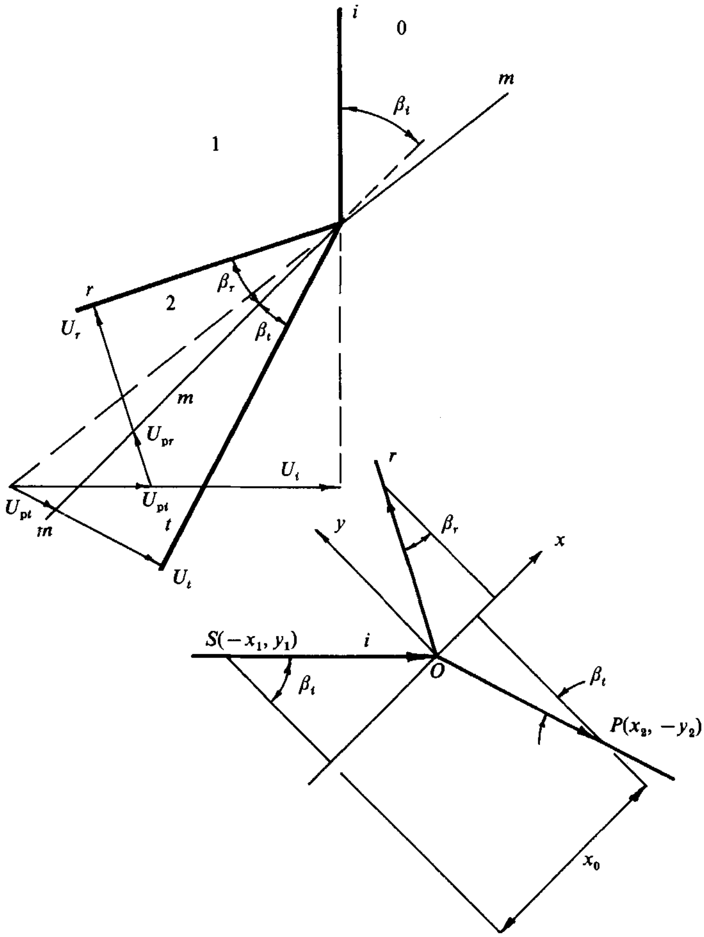

FIGURE 5. Wave diagram and ray-path diagram for a regular refraction with a reflected shock wave.

its ray path to the point P on the transmitted shock, figure 5. The time τ for the point on the ray path to pass from S to P is

$$\tau = \frac{[x_1^2+y_1^2]^{\frac{1}{2}}}{U_i} + \frac{[(x_0-x_1)^2+y_2^2]^{\frac{1}{2}}}{U_t}.$$

Now U_i, U_t, are constants along any ray path for a stationary or pseudostationary system, so the variations that have to be considered are only those of the direction of the ray path. Therefore x_1 is the only variable on the right-hand side of the equation. If τ is to have a stationary value for the path, then

$$\frac{d\tau}{dx_1} = 0 = \frac{x_1}{U_i[x_1^2+y_1^2]^{\frac{1}{2}}} - \frac{(x_0-x_1)}{U_t[(x_0-x_1)^2+y_2^2]^{\frac{1}{2}}}$$

and thus
$$0 = \frac{\sin\beta_i}{U_i} - \frac{\sin\beta_t}{U_t},$$

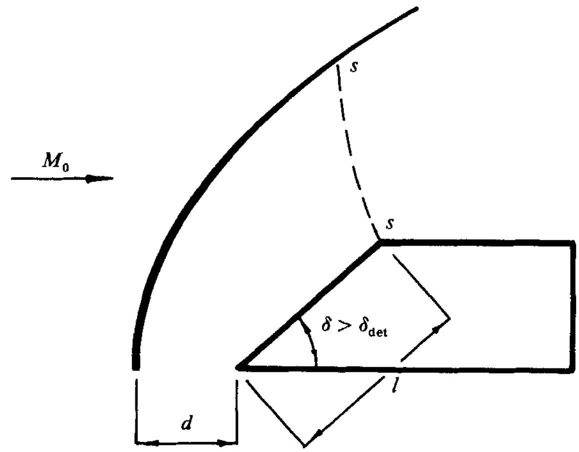

FIGURE 6. A precursor shock wave standing off a blunt wedge.

so we have recovered the refraction law. A second differentiation easily confirms that τ is a minimum. The derivation for the reflected wave, be it either a shock or an expansion, follows without difficulty, and extends the minimum-time principle to include this wave. We omit the details.

When the pressure, temperature, and particle velocity are all constants for the undisturbed media then they are in local thermodynamic equilibrium (LTE). When the wave velocity is constant along a ray path through a medium in LTE the medium behind the wave must also be in LTE so the fact that a stationary or pseudostationary system obeys the refraction law, and therefore that the time has a stationary value along a ray path, implies that the system is in LTE. Let us consider now the minimum-time condition. Since U_i and U_t are constants the ray paths are straight lines before and after the interface. But by the minimum-time condition this also means that U_i and U_t have the maximum values that are compatible with the system boundary condition, namely the driving piston velocity U_{pi}. Furthermore, the shock Mach numbers, M_i and M_t, will also be maxima, and so therefore will the entropy change (production) per unit mass through each shock. Now this is also a necessary condition for a system in LTE to be thermodynamically stable (Henderson 1988). We conclude that the minimum-time principle for a stationary or pseudostationary refracting system implies that the system obeys the refraction law, is in LTE, and satisfies a necessary condition for thermodynamic stability.

6.2. *Bound- and free-precursor shocks*

Suppose there is a wedge of apex angle δ in a supersonic stream of Mach number M_0, and such that δ exceeds the shock detachment angle $\delta > \delta_{det}$, figure 6. The flow along the sloping surface will be subsonic and terminated by a sonic surface *ss* that extends from the wedge corner to the shock. The shock stand-off distance *d* depends on M_0, δ, the properties of the medium, and a boundary lengthscale, such as the length *l* of the sloping surface. Since the shock is at a non-zero distance *d* upstream of the disturbance that produces it (the wedge) we shall say that it is a precursor shock. When the flow is in a steady state then *d* will be constant in time and we shall then say that the shock is a *bound precursor*.

Suppose now that the lengthscale *l* is growing uniformly with time so that the flow

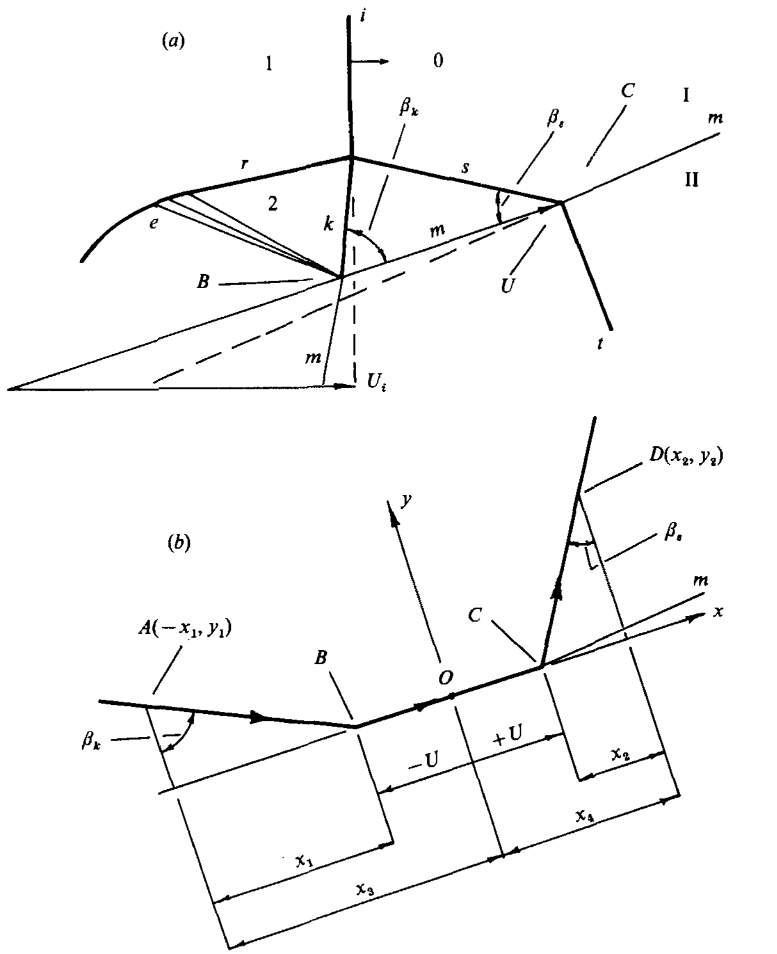

FIGURE 7. Wave diagram and ray path diagram for a free precursor irregular shock-wave refraction.

is pseudostationary (self-similar), then d will also grow uniformly. We shall then say that the shock is a *free precursor*.

6.3. *The refraction law for the free-precursor irregular refraction*

Precursor transmitted shocks are observed when the refraction law is violated at the critical angle β_c, (48). However the law is of course instantly re-established in a different form for the new system (figure 7*a*). For instance, the law is obeyed by the transmitted-side shock-wave pair (*ts*) and separately by the modified shock-expansion-wave pair (*ke*), otherwise these local wave systems would also break up. The wave systems are usually pseudostationary in shock-tube experiments, so *free-precursor* transmitted shocks are observed (Jahn 1956; Abdel-Fattah & Henderson 1978*b*; figure 7*a*).

In a pseudostationary system, the distance BC grows uniformly in time. In order to derive the refraction law, we shall take the origin of coordinates, O, on the disturbed interface, and chosen in such a way that the point C moves away from O with the velocity $+U$ in the positive x-direction, where x lies along the disturbed interface. Similarly, the point B moves away from O at $-U$ in the negative x-direction. Then the refraction law for the side shock s, and the modified shock, k is simply

$$\left.\begin{aligned} U = \frac{U_s}{\sin\beta_s}, \quad -U = \frac{U_k}{\sin\beta_k} \\ \text{or} \quad \frac{U_s}{\sin\beta_s} - U = 0 = \frac{U_k}{\sin\beta_k} + U, \end{aligned}\right\} \tag{53}$$

where U_s, U_k, are the shock-wave velocities and β_s, β_k, their wave angles with respect to the disturbed interface. The law is of course immediately extendable to include the t and e waves.

The minimum-time principle

Constructing the ray-path diagram for the arbitrary point A on the k shock as it proceeds to D on the s shock, we have, for the time between A and D,

$$\tau = \frac{[x_1^2+y_1^2]^{\frac{1}{2}}}{U_k} + \frac{[x_2^2+y_2^2]^{\frac{1}{2}}}{U_s} + \frac{x_3-x_1}{-U} + \frac{x_4-x_2}{U}.$$

In this case there are two independent variables x_1, x_2, the other quantities on the right-hand side are constants. Expanding in a Taylor series about the given path, we have, to second order,

$$\mathrm{d}\tau = \frac{\partial\tau}{\partial x_1}\mathrm{d}x_1 + \frac{\partial\tau}{\partial x_2}\mathrm{d}x_2 + \frac{1}{2!}\left\{\frac{\partial^2\tau}{\partial x_1^2}\mathrm{d}x_1^2 + 2\frac{\partial^2\tau}{\partial x_1\,\partial x_2}\mathrm{d}x_1\,\mathrm{d}x_2 + \frac{\partial^2\tau}{\partial x_2^2}\mathrm{d}x_2^2\right\}.$$

Changing the notation to $\tau_{x_1} \equiv \partial\tau/\partial x_1$, and so on, and completing the square for the second-order terms results in

$$\mathrm{d}\tau = \tau_{x_1}\mathrm{d}x_1 + \tau_{x_2}\mathrm{d}x_2 + \frac{1}{2!}\left\{\frac{1}{\tau_{x_1x_1}}[\tau_{x_1x_1}\mathrm{d}x_1 + \tau_{x_1x_2}\mathrm{d}x_2]^2 + \left[\tau_{x_2x_2} - \frac{\tau_{x_1x_2}^2}{\tau_{x_1x_1}}\right]\mathrm{d}x_2^2\right\}.$$

Evaluating the derivatives

$$\frac{\partial\tau}{\partial x_1} = \frac{x_1}{U_k[x_1^2+y_1^2]^{\frac{1}{2}}} + \frac{1}{U}, \tag{54}$$

$$\frac{\partial\tau}{\partial x_2} = \frac{x_2}{U_s(x_2^2+y_2^2]^{\frac{1}{2}}} - \frac{1}{U}, \tag{55}$$

$$\frac{\partial^2\tau}{\partial x_1^2} = \frac{1}{U_k}\frac{y_1^2}{[x_1^2+y_1^2]^{\frac{3}{2}}} > 0 \tag{56}$$

$$\frac{\partial^2\tau}{\partial x_2^2} = \frac{1}{U_s}\frac{y_2^2}{[x_2^2+y_2^2]^{\frac{3}{2}}} > 0, \tag{57}$$

$$\frac{\partial^2\tau}{\partial x_1\,\partial x_2} = 0; \tag{58}$$

thus

$$\left(\frac{\partial^2\tau}{\partial x_1^2}\right)^{-1}\left\{\frac{\partial^2\tau}{\partial x_1^2}\frac{\partial^2\tau}{\partial x_2^2} - \left(\frac{\partial^2\tau}{\partial x_1\,\partial x_2}\right)^2\right\} > 0. \tag{59}$$

The first-order derivatives must be zero if τ is to have a stationary value, but then (54) and (55) give

$$\frac{\sin\beta_k}{U_k}+\frac{1}{U}=0=\frac{\sin\beta_s}{U_s}-\frac{1}{U},$$

which gives again the refraction law (53). The stationary value for τ is clearly a minimum by inspection of (56) to (59).

The principle may now be asserted. *An arbitrary point on a shock or an expansion wave that is propagating in a stationary or pseudostationary system will trace out a ray path of minimum time between any two points on the path irrespective of the medium in which the wave is moving.*

The principle is applicable to all known stationary and pseudostationary regular and irregular refracting systems, and it is easily extended to regular and Mach reflection by allowing the second medium to approach the rigid limit. The principle is more general than for acoustic refraction because it takes into account the deflection of the interface by the waves and the fact that at least some of the waves are propagating in media that are themselves in motion.

7. Conclusions

1. The wave impedance determines the nature and the intensity of the reflected and transmitted waves, and the fractions of the energy and the power that are reflected and transmitted. In particular:

(*a*) The sign of the transmitted wave impedance Z_t is always the same as that of the incident wave Z_i, so if the incident wave is a shock so also will be the transmitted wave.

(*b*) (i) When $|Z_t| > |Z_i|$ then $R > 0, T > 0$, and the reflected and transmitted waves are both shocks.

(ii) When $|Z_t| = |Z_i|$, then, $R = 0$, $T_\pi = T_I = T = 1$, and there is a transmitted shock, but no reflected wave. This is the condition for total transmission.

(iii) When $|Z_t| < |Z_i|$, $R < 0, T > 0$, the transmitted wave is again a shock but the reflected wave is an expansion.

(*c*) (i) *At the acoustic limit*, where the wave impedances correspond to the acoustic impedances, R, (29), and T, (31), reduce to the symmetric acoustic formulas and display the principle of acoustic reciprocity. However the principle cannot be applied to shock waves because (29) and (31) are not symmetrical, except at the limit.

(ii) *At the rigid limit*, $|Z_t| \to \infty$, $R_I \to R \to Z_r/Z_i > 0$, $T \to 1+Z_r/Z_i$, $T_\pi \to T_I \to 0$, so a shock penetrates a rigid body with an increase in intensity, but with no transmitted energy.

(iii) *At the compliant limit*, $Z_t \to 0$, $R \to -1$, $T_\pi \to T_I \to T \to 0$, there is no transmitted shock and no transmitted energy; this is the condition of total internal reflection, or pressure release. The reflected wave is an expansion.

2. All of these conclusions may be extended to oblique shocks by a suitable definition of oblique-wave impedance and intensity, equations (49) and (50).

3. The fundamental law of refraction expresses the condition that all the waves meeting at a point must propagate at the same velocity along any trajectory path that passes through the point if the wave system is to be stationary or pseudostationary.

4. A wave will be bent or refracted whenever it encounters a change in the refractive index n, (45), of the medium in which it is propagating. This is equivalent

to a change in the direction and, or, the magnitude of the wave velocity vector U. in particular:

(*a*) When $n < 1$ and $\beta_i < \beta_c$, where β_c, (48), is the *angle of intromission*, then the shock is bent towards the normal to the interface (its slope is increased by refraction), $\beta_i < \beta_t$.

(*b*) When $n = 1$, there is no refraction even though the wave impedance may change, (45).

(*c*) When $n > 1$, the shock is bent away from the normal and becomes less steep, $\beta_i > \beta_t$.

(*d*) When $\beta_i > \beta_c$, then $\cos\beta_t$, (47) is pure imaginary. When this happens to a regular refraction the system breaks up, violating the refraction law, and resulting in the appearance of an irregular wave system with precursor shocks. The refraction law is instantly re-established for the new system.

5. Stationary and pseudostationary wave systems obey a minimum-time principle, that is, the propagation time is a minimum between any two points along an arbitrary wave ray path. The principle is a generalization of Fermat's Principle for acoustic waves, and implies that the system is in LTE and that its production of entropy is a maximum, which is a necessary condition for thermodynamic stability. The stationary time condition implies the refraction law.

Appendix. Regular refraction of a plane shock wave in a perfect gas where the reflected wave is a shock, RRR

The exact polynomial equation

The equation is of degree 12 and of the form

$$\sum_{i=1}^{9} R_i T_i = 0,$$

where R_i and T_i are polynomials each of degree 6, so there are 18 polynomial factors in all. However twelve of them are simple multiples of the other six. If

$$x \equiv P_t/P_0$$

is the dependent variable, and

$$A \equiv \cos\delta_0; \qquad \alpha_1 \equiv P_0/P_1,$$

$$b_0 \equiv 1+\gamma_i M_0^2; \qquad b_1 \equiv 1+\gamma_i M_1^2,$$

$$b_t \equiv 1+\gamma_t M_t^2;$$

$$c_i \equiv \frac{\gamma_i-1}{\gamma_i+1}; \qquad c_t \equiv \frac{\gamma_t-1}{\gamma_t+1},$$

$$d_0 \equiv \frac{2\gamma_i}{\gamma_i+1} M_0^2 - \frac{\gamma_i-1}{\gamma_i+1}; \quad d_1 = \frac{2\gamma_i}{\gamma_i+1} M_1^2 - \frac{\gamma_i-1}{\gamma_i+1},$$

$$d_t = \frac{2\gamma_t}{\gamma_t+1} M_t^2 - \frac{\gamma_t-1}{\gamma_t+1}; \quad M_t = M_0\left[\frac{\gamma_i\,\mu_t}{\gamma_t\,\mu_i}\right]^{\frac{1}{2}},$$

where M_0, M_1, M_t are the free-stream Mach numbers upstream of the i, r, and t shocks respectively, and μ_i, μ_t are the molecular weights of the incident and transmitting media, then

$$
\begin{aligned}
R_1 &\equiv (b_t-x)^4(c_t+x)^2; & T_1 &\equiv (b_1-a_1x)^4(c_i+a_1x)^2,\\
R_2 &\equiv -2A^2T_6; & T_2 &\equiv T_1,\\
R_3 &\equiv -2A^2R_1; & T_3 &\equiv (b_1-a_1x)^2(c_i+a_1x)(a_1x-1)^2(d_1-a_1x),\\
R_4 &\equiv A^4T_1; & T_4 &\equiv T_9,\\
R_5 &\equiv A^4R_1; & T_5 &\equiv R_9,\\
R_6 &\equiv -2(A^4+4A^2+1)T_3; & T_6 &\equiv (b_t-x)^2(c_t+x)(x-1)^2(d_t-x),\\
R_7 &\equiv -2A^2T_3; & T_7 &\equiv T_9,\\
R_8 &\equiv R_2 \equiv -2A^2T_6; & T_8 &\equiv T_5 \equiv R_9,\\
R_9 &\equiv (a_1x-1)^4(d_1-a_1x)^2; & T_9 &\equiv (x-1)^4(d_t-x)^2.
\end{aligned}
$$

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